South East Asian J. of Mathematics and Mathematical Sciences
Vol. 21, No. 2 (2025), pp. 25-42

DOI: 10.56827/SEAJMMS.2025.2102.2 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

SPECTRUM OF THE GENERALIZED ZERO-DIVISOR GRAPHS

Krishnat Masalkar, Anita Lande and Anil Khairnar

Department of Mathematics,
Abasaheb Garware College,
Pune - 411004, Maharashtra, INDIA

E-mail : krishnatmasalkar@gmail.com, anita7783@gmail.com,
ask.agc@mespune.in

(**Received:** *Oct. 24, 2024* **Accepted:** *Aug. 26, 2025* **Published:** *Aug. 30, 2025)*

Abstract: The generalized zero-divisor graph of a ring R, denoted by $\Gamma'(R)$, is a simple (undirected) graph with a vertex set consisting of all nonzero zero-divisors in R, and two distinct vertices x and y are adjacent if $x^n y = 0$ or $y^n x = 0$, for some positive integer n. If $R = \prod_{i=1}^{k} R_i$ is a direct product of finite commutative local rings R_i with $|R_i| = p_i^{\alpha_i}$, then we express $\Gamma'(R)$ as a H-generalized join of a family $\mathcal{F}$ of a complete graph and null graphs, where H is a graph obtained from $\Gamma'(S^k)$ by contraction of edges of all nonzero nilpotents at a single vertex $\mathbf{0}$, and $S = \{0, 1, 2\}$ is a multiplicative submonoid of a ring $\mathbb{Z}_4$. Also, we prove that the adjacency spectrum of $\Gamma'(R)$ is $\left\{(-1)^{(\beta-1)}, 0^{(\gamma-3^k+2^k+1)}\right\} \cup \sigma(NA(H))$, where β is the number of nonzero nilpotent elements, γ is the number of non-nilpotent zero-divisors in R and N is a diagonal matrix whose rows (columns) are indexed with vertices $e \in \Gamma'(H)$ with e^{th} diagonal entry is the cardinality of e^{th} graph in the family $\mathcal{F}$.

Keywords and Phrases: Eigenvalue, generalized zero-divisor graph, complete graph, regular graph, adjacency matrix.

2020 Mathematics Subject Classification: 05C25, 05C50.

1. Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The *adjacency matrix* of graph G denoted by $A(G) = [a_{ij}]$, is a matrix whose rows (columns) are indexed with vertices of G, and $a_{ij} = 1$, if i^{th} and j^{th} vertices are adjacent in G, and $a_{ij} = 0$ otherwise. The *adjacency spectrum* of a graph G, denoted by $\sigma_A(G)$, is a spectrum of the matrix $A(G)$. If K_n is complete graph on n vertices and $\overline{K}_n$ is null graph on n vertices, then $\sigma_A(K_n) = \{(-1)^{(n-1)}, (n-1)^{(1)}\}$ and $\sigma_A(\overline{K}_n) = \{0^{(n)}\}$. Let H be a graph with vertex set $[n] = \{1, 2, \ldots, n\}$ and $\mathcal{F} = \{G_1, G_2, \ldots, G_n\}$ be a family of r_i-regular graphs G_i with $|G_i| = k_i$. If G is a graph obtained by replacing i by G_i and every vertex of G_i is joined to every vertex of G_j if and only if i and j are adjacent in H, then G is called as *H-generalized join* of the family of graphs $\mathcal{F}$, we write it as $G = \bigvee_H G_i$. Recall the following result from [17].

Theorem 1.1. [17] *Let H be a graph with vertex set $[n] = \{1, 2, \ldots, n\}$ and $\{G_1, G_2, \ldots, G_n\}$ be a family of r_i-regular graphs G_i with $|G_i| = k_i$. If $G = \bigvee_H G_i$, then*

$$\sigma_A(G) = \sigma(diag(k_1, k_2, \ldots, k_n)A(H)) \cup \bigcup_{i=1}^{n} (\sigma_A(G_i) \setminus \{r_i\}). \tag{1.1}$$

There is an interplay between the adjacency spectrum and structural properties of a graph, see [8].

A mapping $*$ on an associative ring is called an *involution* if for all $x, y \in R$: $(x + y)^* = x^* + y^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$. A ring equipped with involution $*$ is called a *$*$-ring*. The concept of the zero-divisor graph of a commutative ring was first introduced by Beck in 1988, [5]. He defined the zero-divisor graph of a commutative ring R, with a vertex set of all elements of R, and two distinct vertices x and y are adjacent if $xy = 0$. Anderson and Livingston [1] (1999), introduced a zero-divisor graph for a commutative ring R, denoted by $\Gamma(R)$ as a simple (undirected) graph, with a vertex set $Z^*(R)$ the set of all nonzero zero-divisors in R, and two distinct vertices x and y are adjacent in $\Gamma(R)$ if $xy = 0$. Patil and Waphare [15] introduced a zero-divisor graph of a $*$-ring R. Kumbhar et al. [9] introduced the strong zero-divisor graph of $*$-rings. In [11], authors introduced a generalized zero-divisor graph of a $*$-ring R, denoted by $\Gamma'(R)$. They associated a simple (undirected) graph with the vertex set $Z^*(R)$, and two distinct vertices x and y are adjacent if $x^n y^* = 0$ or $y^n x^* = 0$, for some positive integer n. If R is a commutative ring, then the identity mapping is the only involution on R. Hence, the *generalized zero-divisor graph* of a commutative ring R is a simple graph with vertex set $Z^*(R)$, and two distinct vertices x and y are adjacent if $x^n y = 0$ or $y^n x = 0$, for some positive integer n. Recently, in [12], authors studied the spectrum of the generalized zero-divisor graph of the ring $\mathbb{Z}_{p^\alpha q^\beta}$ for distinct primes p, q and positive integers α, β. John D. Lagrange [10] shows that all eigenvalues of $\Gamma(\mathbb{Z}_2^k)$ are eigenvalues of Pascal-type matrices. The study of the spectrum of zero-divisor graphs is explored in [6, 13, 14, 16]. Readers refer to [2, 3, 8] for concepts in zero-divisor graphs, ring theory, and graph theory, respectively.

In [18], the authors considered a finite reduced ring R_n with n maximal ideals. The class of rings R_n contains the Boolean rings as a subclass. They studied the eigenvalues of finite reduced rings in terms of the eigenvalues of Boolean rings using equitable partition. Let R be a direct product of local commutative rings with unity. In this paper, we study eigenvalues of $\Gamma'(R)$ in terms of the eigenvalues of $\Gamma'(S^k)$, where S is a multiplicative submoid of the ring $\mathbb{Z}_4$. In the second section, we study the elementary structural properties of the generalized zero-divisor graph of rings, and we associate a generalized zero-divisor graph to a multiplicative submonoid. In the third section, we expressed the adjacency spectrum of $\Gamma'(S^k)$, where $S = \{0, 1, 2\}$, which is a submonoid of the ring $\mathbb{Z}_4$ with respect to multiplication. In the fourth section, for any finite commutative ring, we express the graph $\Gamma'(R)$ as a generalized join of a complete graph and null graphs and obtain its adjacency spectrum. We find the multiplicities of eigenvalues 0 and -1 of $\Gamma'(R)$ and express the remaining eigenvalues in terms of eigenvalues of $\Gamma'(S^k)$, where R is a ring which is a direct product of finite commutative local rings with unity. In the fifth section, as an application, we give illustrative examples to find the adjacency spectrum of $\Gamma'(R)$, where R is a direct product of finite commutative local rings.

2. The generalized zero-divisor graph $\Gamma'(R)$

Let R be a commutative ring. The *generalized zero-divisor graph* $\Gamma'(R)$ is a simple (undirected) graph with vertex set the set of all nonzero zero-divisors in R and two distinct vertices x and y are adjacent if $x^n y = 0$ or $xy^n = 0$, for some positive integer n. We use the exact definition to define the generalized zero-divisor graph of a finite commutative monoid with respect to multiplication.

It is clear that $\Gamma(R)$ and $\Gamma'(R)$ have the same set of vertices, and if two vertices x and y are adjacent in $\Gamma(R)$, then they are adjacent in $\Gamma'(R)$ but not conversely. In [1], Anderson et al. proved that for a commutative ring R, $\Gamma(R)$ is connected and $diam(\Gamma(R)) \leq 3$. We have $\Gamma'(R)$ is connected and $diam(\Gamma'(R)) \leq 3$.
The following are elementary properties of $\Gamma'(R)$.

Remark 2.1. *Let R be a finite commutative ring.*

1. *If x is a nonzero nilpotent element in R, then it is adjacent to all the other*

vertices in $\Gamma'(R)$.

2. *If two vertices* x *and* y *are adjacent in* $\Gamma'(R)$, *then for any two positive integers* i, j, *the vertices* x^i *and* y^j *are also adjacent in* $\Gamma'(R)$.
3. *If* $R \neq \mathbb{Z}_2 \times \mathbb{Z}_2$ *and* x *is adjacent to all the other vertices, then* x *is a nilpotent element in* R.
4. *For a reduced ring* R, $\Gamma'(R) \simeq \Gamma(R)$.

The following lemma gives the condition under which $\Gamma'(R)$ is a complete graph.

Lemma 2.2. *Let* R *be a finite commutative ring. Then* $\Gamma'(R)$ *is a complete graph if and only if* R *is a local ring. In particular, if the number of nonzero nilpotent elements in a local ring* R *is* m, *then* $\Gamma'(R) = K_m$. *Further,* $\Gamma'(\mathbb{Z}_{p^n}) = K_{p^{n-1}-1}$.

Proof. Let R be a finite commutative ring. R is a local ring if and only if $Z^*(R)$ is the set of all nonzero nilpotent elements in R. Thus $\Gamma'(R)$ is a complete graph. Further, if $R = \mathbb{Z}_{p^n}$, then $Z^*(R) = \{0, p, 2p, \ldots, p^{n-1}\}$. Therefore $\Gamma'(R) = K_{p^{n-1}-1}$.

Let R be a finite commutative ring of size n and n has a prime factorization $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$. Then R is direct product of finite local rings $R_{p_i^{\alpha_i}}$ of order $p_i^{\alpha_i}$, for all $i = 1, 2, \ldots, k$. In a local ring, every element is nilpotent or a unit. Every nonzero nilpotent element of a ring R is adjacent to any other vertex, since if x is a nonzero nilpotent element in R, then $x^n y = 0$, for some positive integer n and every $y \in V(\Gamma'(R)$. The extended generalized zero-divisor graph $\Gamma'_e(R)$ is a graph with a vertex set R and any two vertices x, y are adjacent if $x^n y = 0$ or $xy^n = 0$, for some positive integer n.

3. Adjacency spectrum of $\Gamma'(S^k)$, $S = \{0, 1, 2\}$

Definition 3.1. [7] *Let* $G = (V, E)$ *be a graph. A partition* $\Pi = X_1 \cup X_2 \cup \cdots \cup X_k$ *of* V *is said to be an equitable partition if there are numbers* q_{ij}, $i, j \in [k]$ *such that every vertex in* X_i *is adjacent to exactly* q_{ij} *vertices in* X_j.

Let $G = (V, E)$ be a graph and $V = X_1 \cup X_2 \cup \cdots \cup X_k$ with $X_i \cap X_j = \phi$, for all $i \neq j \in [n]$. Suppose that every vertex in X_i is adjacent with exactly q_{ij} vertices in X_j for all $i, j \in [n]$ and $P = [p_{ij}]$ be a matrix whose rows are indexed by vertices in V and columns are indexed by sets $X_1, X_2, \ldots, X_n$, where

$$p_{ij} = \begin{cases} 1 & \text{if } v_i \in X_j \\ 0 & \text{otherwise.} \end{cases}$$

Then $Q = [q_{ij}]$ is called the *quotient matrix*. Let α be a set of indices with exactly one vertex from each X_i, and α^c is the complement of α. For any matrix M,

$M[\alpha : \beta]$ represents a submatrix whose row indices are given by α and column indices are given by β. Let $M[: \beta]$ represent the submatrix with all row and column indices given by β. Recall the following theorem from [19].

Theorem 3.2. [19] *Let $A(G)$ be the adjacency matrix of a graph G, and let Q be the quotient matrix corresponding to an equitable partition $\Pi = \{X_1, X_2, \dots, X_k\}$. Let P be the characteristic matrix of Π and let α be an index set that contains exactly one element from each X_i, $i \in [k]$.*

$$\sigma_A(G) = \sigma(Q) \cup \sigma(Q^*), \tag{3.1}$$
$$\text{where } Q^* = A(G)[\alpha^c : \alpha^c] - P[\alpha^c :]A(G)[\alpha : \alpha^c].$$

Observe that Theorem 1.1 is a particular case of Theorem 3.2, since the vertex sets of a family of graphs in the generalized join graph of regular graphs form an equitable partition. Next, recall the generalized Cauchy interlacing theorem and some of its consequences [8].

Theorem 3.3. [8] *Eigenvalues of a real symmetric matrix interlace with those of its principal submatrices. That is, if $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are eigenvalues of a real symmetric matrix M and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m$ are eigenvalues of its principal submatrix of size m then*

$$\lambda_i \leq \mu_i \leq \lambda_{n-m+i}, \quad \text{for} \quad i = 1, 2, \dots, m. \tag{3.2}$$

The set $S = \{0, 1, 2\}$ is monoid of the ring $\mathbb{Z}_4$ with respect to multiplication. That is, S is a subset of a ring $\mathbb{Z}_4$ which is closed with respect to multiplication. One can consider a zero-divisor graph on a subset of a ring. We use set S and the graph $\Gamma'_e(S)$ to study the graph $\Gamma'_e(R)$ in subsequent results. The adjacency matrix of an extended generalized zero-divisor graph (which is simple, so that it has no loops) $\Gamma'_e(S)$ is

$$A(\Gamma'_e(S)) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \mathbf{1}_3 - I_3, \text{ where } \mathbf{1_n} \text{ is a matrix of all ones of size n.}$$

It is clear that

$$\sigma(A(\Gamma'_e(S)) = \left\{2^{(1)}, (-1)^{(2)}\right\}. \tag{3.3}$$

Recall that for any two graphs G_1 and G_2, $A(G_1 \otimes G_2) = A(G_1) \otimes A(G_2)$. Also for any two square matrices M_1 and M_2 of same size, $\sigma(M_1 \otimes M_2) = \sigma(M_1).\sigma(M_2)$.

Note that $\sigma(M_1)$ and $\sigma(M_2)$ are multisets and $\sigma(M_1).\sigma(M_2)$ is a multiset which is obtained by taking the product of each element in $\sigma(M_1)$ with each element in $\sigma(M_2)$, with counting multiplicities. Hence, we have the following.

$$\begin{aligned}\sigma(A(\Gamma'_e(S^k))) &= \sigma(A(\otimes^k(\Gamma'_e(S))) = (\sigma(A(\Gamma'_e(S)))^k \\ &= \left\{\left((-1)^{(j)}(2)^{(k-j)}\right)^{\binom{k}{j}} : j = 0, 1, 2, \dots, k\right\}.\end{aligned} \tag{3.4}$$

A set $X = \{0,2\}^k \setminus \{0\}^k$ is a set of all nonzero nilpotent elements in S^k and $Y = S^k \setminus \left(\{0,2\}^k \cup \{1\}^k\right)$ is a set of all non-nilpotent zero-divisors in S^k. Then $|X| = 2^k - 1$ and $|Y| = 3^k - 2^k - 1$. Let $\Gamma'(S^k \setminus \{0\}^k)$ be subgraph of $\Gamma'_e(S^k)$ on vertices $S^k \setminus \{0\}^k$, $C_k = A(\Gamma'(S^k \setminus \{0\}^k))$ and $D_k = A(\Gamma'(S^k))$. Then the adjacency matrix of the graph $\Gamma'_e(S^k)$ with respect to vertex ordering $\left\{\{0\}^k, \{1\}^k, X, Y\right\}$ is given by

$$A(\Gamma'_e(S^k)) = \begin{bmatrix} 0 & 1 & A(\{0\}^k, X) & A(\{0\}^k, Y) \\ 1 & 0 & A(\{1\}^k, X) & A(\{1\}^k, Y) \\ A(X, \{0\}^k) & A(X, \{1\}^k) & A(X, X) & A(X, Y) \\ A(Y, \{0\}^k) & A(Y, \{1\}^k) & A(Y, X) & A(Y, Y) \end{bmatrix}, \tag{3.5}$$

where $A(X_1, X_2)$ is an adjacency matrix between vertex sets X_1 and X_2. Therefore, we have

$$A(\Gamma'_e(S^k)) = \begin{bmatrix} 0 & 1 & 1_{1,2^k-1} & 1_{1,3^k-2^k-1} \\ 1 & 0 & 1_{1,2^k-1} & 0_{1,3^k-2^k-1} \\ 1_{2^k-1,1} & 1_{2^k-1,1} & 1_{2^k-1,2^k-1} - I_{2^k-1} & 1_{2^k-1,3^k-2^k-1} \\ 1_{3^k-2^k-1,1} & 0_{3^k-2^k-1,1} & 1_{3^k-2^k-1,2^k-1} & M_{3^k-2^k-1,3^k-2^k-1} \end{bmatrix}, \tag{3.6}$$

$$C_k = \begin{bmatrix} 0 & 1_{1,2^k-1} & 0_{1,3^k-2^k-1} \\ 1_{2^k-1,1} & 1_{2^k-1,2^k-1} - I_{2^k-1} & 1_{2^k-1,3^k-2^k-1} \\ 0_{3^k-2^k-1,1} & 1_{3^k-2^k-1,2^k-1} & M_{3^k-2^k-1,3^k-2^k-1} \end{bmatrix}, \tag{3.7}$$

$$D_k = \begin{bmatrix} 1_{2^k-1,2^k-1} - I_{2^k-1} & 1_{2^k-1,3^k-2^k-1} \\ 1_{3^k-2^k-1,2^k-1} & M_{3^k-2^k-1,3^k-2^k-1} \end{bmatrix}, \tag{3.8}$$

where $M_{3^k-2^k-1,3^k-2^k-1} = A(Y,Y)$.
By theorem 3.3, eigenvalues of C_k interlace those of D_k and eigenvalues of $A(\Gamma'_e(S^k))$ interlace those of C_k. That is, if $\lambda_1 \leq \cdots \leq \lambda_{3^k}$ are eigenvalues of $A(\Gamma'_e(S^k))$, $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{3^k-2}$ are eigenvalues of C_k and $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_{3^k-2}$ are

eigenvalues of D_k, then

$$\lambda_i \leq \mu_i \leq \lambda_{2+i} \quad \text{for} \quad i = 1, 2, \ldots, 3^k - 2 \tag{3.9}$$

$$\mu_i \leq \nu_i \leq \mu_{1+i} \quad \text{for} \quad i = 1, 2, \ldots, 3^k - 3. \tag{3.10}$$

If k is even, then the eigenvalues of $\Gamma'_e(S^k)$ can be written as

$$\underbrace{-2^{k-1} = \cdots = -2^{k-1}}_{\binom{k}{1} \text{ times}} < \underbrace{-2^{k-3} = \cdots = -2^{k-3}}_{\binom{k}{3} \text{ times}} < \underbrace{-2^{k-5} = \cdots = -2^{k-5}}_{\binom{k}{5} \text{ times}} < \cdots < \tag{3.11}$$

$$\underbrace{-2 = \cdots = -2}_{\binom{k}{k-1} \text{ times}} < 1 < \underbrace{2^2 = \cdots = 2^2}_{\binom{k}{2} \text{ times}} < \underbrace{2^4 = \cdots = 2^4}_{\binom{k}{4} \text{ times}} < \cdots < \underbrace{2^{k-2} = \cdots = 2^{k-2}}_{\binom{k}{k-2} \text{ times}} < 2^k.$$

If k is odd, then the eigenvalues of $\Gamma'_e(S^k)$ can be written as

$$\underbrace{-2^{k-2} = \cdots = -2^{k-2}}_{\binom{k}{2} \text{ times}} < \underbrace{-2^{k-4} = \cdots = -2^{k-4}}_{\binom{k}{4} \text{ times}} < \cdots < \tag{3.12}$$

$$\underbrace{-2^2 = \cdots = -2^2}_{\binom{k}{k-2} \text{ times}} < -1 < \underbrace{2 = \cdots = 2}_{\binom{k}{1} \text{ times}} < \underbrace{2^3 = \cdots = 2^3}_{\binom{k}{3} \text{ times}} < \cdots < \underbrace{2^{k-1} = \cdots = 2^{k-1}}_{\binom{k}{k-1} \text{ times}} < 2^k.$$

By equations (3.9) to (3.12), we state the following theorem.

Theorem 3.4. *Let $S = \{0, 1, 2\}$ be a monoid of the ring $\mathbb{Z}_4$ with respect to multiplication and k be a positive integer. Then:*

If k is even

$$\sigma_A(\Gamma'(S^k)) = \left\{((-1)^j 2^{k-j})^{(\binom{k}{j}-2)} : j = 1, 2, \ldots, k-1\right\} \cup \{\mu_1, \mu_2, \ldots, \mu_{2k-2}\}, \tag{3.13}$$

where $-2^{k-i} < \mu_i \leq \mu_{i+1} < -2^{k-i-2}$, *for* $i = 1, 3, 5, \ldots, k-1$,
$-2 < \mu_{k-1} \leq \mu_k < 4$, $2^i < \mu_{i+k-1} \leq \mu_{i+k} < 2^{i+2}$, *for* $i = 2, 4, \ldots, k-2$.

If k is odd

$$\sigma_A(\Gamma'(S^k)) = \left\{((-1)^j 2^{k-j})^{(\binom{k}{j}-2)} : j = 1, 2, \ldots, k-1\right\} \cup \{\mu_1, \mu_2, \ldots, \mu_{2k-2}\}, \tag{3.14}$$

where $-2^{k-i} < \mu_{i-1} \leq \mu_i < -2^{k-i-2}$, *for* $i = 2, 4, \ldots, k-1$, *and*
$-4 < \mu_{k-1} \leq \mu_k < 2$, $2^i < \mu_{i+k} \leq \mu_{i+k+1} < 2^{i+2}$, *for* $i = 1, 3, \ldots, k-2$.

To find these unknown eigenvalues $\mu_1, \mu_2, \ldots, \mu_{2k-2}$, we use the concept of equitable partitions and quotient matrices of the equitable partitions. We find an equitable partition of the vertex set of $\Gamma'(S^k)$ as below.
For each $(i,j) \in [k-1] \times \{0,1,\ldots,k-i\}$, define

$$X_0 = \{0,2\}^k \setminus \{0\}^k \tag{3.15}$$

$$X_{i,j} = \left\{x \in \{0,1,2\}^k : x \text{ has } \; i \; 1's \text{ and } j \; 0's\right\}. \tag{3.16}$$

Let m be the number of $X_{i,j}{}'s$. Then $m = 1+2+\cdots+k-1+k = \frac{(k+1)k}{2}$. We list these sets according to the dictionary order on $\{(i,j) \colon i \in [k-1], j \in \{0,1,\ldots,k-i\}\}$ as below

$$\Pi_m = X_0 \cup \left\{\begin{array}{ccccc} X_{1,0}, & X_{1,1}, & \ldots, & X_{1,k-2}, & X_{1,k-1} \\ X_{2,0}, & X_{2,1}, & \ldots & X_{2,k-2}, & \\ \vdots & \vdots & & & \\ X_{k-1,0} & X_{k-1,1} & & & \end{array}\right\}. \tag{3.17}$$

In the following result, we prove that the equation (3.17) forms an equitable partition.

Theorem 3.5. *The family of sets Π_m in the equation* (3.17) *forms an equitable partition of the vertex set of the graph $\Gamma'(S^k)$. Every vertex in $X_{p,q}$ is adjacent to exactly $L_{(p,q),(r,s)}$ number of vertices in $X_{r,s}$, where*

$$L_{(p,q),(r,s)} = \binom{k-p}{r}\binom{k-p-r}{s-p} + \binom{k-p}{r}\binom{k-r}{s} - \binom{k-p-r}{s-p}. \tag{3.18}$$

The quotient matrix associated with the equitable partition Π_m is $m \times m$ matrix given by

$$Q_m = [L_{(p,q),(r,s)}]_{m\times m}, \quad \text{and} \quad \sigma(Q_m) \subseteq \sigma_A(\Gamma'(S^k)). \tag{3.19}$$

Proof. Every vertex in X_0 is adjacent to every other vertex, since it is nilpotent. Every vertex in X_0 is adjacenct to $L_{(0,0),(r,s)}$ number of vertices in $X_{r,s}$. Also, every vertex in X_0 is adjacent to all the other $2^k - 1$ vertices in X_0. Therefore $L_{(0,0),(0,0)} = 2^k - 1$.
Fix sets $X_{p,q}$ and $X_{r,s}$ in Π_m. Let $x \in X_{p,q}$. We will show that x is adjacent to exactly $L_{(p,q),(r,s)}$ number of vertices in $X_{r,s}$. If $y \in X_{r,s}$ is adjacent to x, then $x^2y = 0$ or $xy^2 = 0$. Fix $x^2y = 0$. Then $x_t = 1$ implies $y_t = 0$. Hence $p \le s$ and $r \le k-p$. There are $\binom{k-p}{r}$ choices for $1's$ in y and $\binom{k-p-r}{s-p}$ choices for $0's$ of y .

Hence number of vertices $y \in X_{r,s}$ such that $x^2y = 0$ is $\binom{k-p}{r}\binom{k-p-r}{s-p}$. Now assume that $xy^2 = 0$, then we have $x_t = 1$, which implies $y_t \in \{0, 2\}$. Hence $r \leq k - p$. So counting $1's$ has $\binom{k-p}{r}$ choices and $0's$ has $\binom{k-r}{s}$ choices. Therefore number of $y \in X_{r,s}$ which satisfy $xy^2 = 0$ is $\binom{k-p}{r}\binom{k-r}{s}$. Also, if $x^2y = 0 = xy^2$, then $x_t = 1$ imply that $y_t = 0$ and $x_t = 0$, which gives $y_t = 1$ and hence $p \leq s$ and $q \leq r$. So counting choices for 0's of y, we get $\binom{k-p-r}{s-p}$ number of $y's$ in $X_{r,s}$ which satisfy $x^2y = xy^2 = 0$. Therefore by inclusion-exclusion principle, the number of vertices $y \in X_{r,s}$ which are adjacent to x is $L_{(p,q),(r,s)}$ given by equation (3.18), and the quotient matrix associated to the partition Π_m is given by equation (3.19).

4. Spectrum of $\Gamma'(R)$

Let R be a finite commutative local ring. We define $\phi(R)$ to be the number of units in R and $f(R)$ to be the number of nonzero zero-divisors in R. Since in a finite commutative local ring R, every nonzero zero-divisor is a nilpotent element, hence $f(R) = |R| - \phi(R) - 1$. Let $U(R)$ denotes the set of units in R. Then $U(R_1 \times R_2 \times \cdots \times R_n) = U(R_1) \times U(R_2) \times \cdots \times U(R_n)$. Hence $\phi(R_1 \times R_2 \times \cdots \times R_n) = \phi(R_1)\phi(R_2)\cdots\phi(R_n)$.

The following result gives the adjacency spectrum of the extended generalized zero-divisor graph $\Gamma'_e(R)$ for a finite commutative local ring R.

Theorem 4.1. *Let R be a finite commutative local ring with m nilpotent elements and n units. Then*

$$\sigma_A(\Gamma'_e(R)) = \left\{ 0^{(n-1)}, (-1)^{(m-1)}, \frac{m-1+\sqrt{(m-1)^2+4mn}}{2}, \frac{m-1-\sqrt{(m-1)^2+4mn}}{2} \right\}. \tag{4.1}$$

Proof. Let X be the set of nilpotent elements and Y be the set of units in a ring R. Then $|X| = m$ and $|Y| = n$. The adjacency matrix of $\Gamma'_e(R)$ with respect to ordering of vertices $\{X, Y\}$ is

$$A(\Gamma'_e(R)) = \begin{bmatrix} \mathbf{1}_m - I_m & \mathbf{1}_{m,n} \\ \mathbf{1}_{n,m} & 0_n \end{bmatrix}.$$

The nullity of A is $n - 1$ and the nullity of $A + I_{m+n}$ is $m - 1$. Therefore

$$\{0^{(n-1)}, (-1)^{(m-1)}\}\} \subseteq (\Gamma'_e(R)).$$

Also, $\left[\begin{array}{c|c} \mathbf{1}_m - I_m & \mathbf{1}_{m,n} \\ \hline \mathbf{1_{n,m}} & 0_n \end{array}\right]$ is an equitable partition of the matrix $A(\Gamma'_e(R))$ and its quotient matrix is $Q = \begin{bmatrix} m-1 & n \\ m & 0 \end{bmatrix}$. Therefore,

$$\sigma_A(\Gamma'_e(R)) = \{0^{(n-1)}, (-1)^{(m-1)}, \sigma(Q)\} = \{0^{(n-1)}, (-1)^{(m-1)}, \lambda_1, \lambda_2\},$$

where λ_1, λ_2 are roots of equation $x^2 - (m-1)x - mn = 0$.
That is, $\lambda_1 = \frac{m-1+\sqrt{(m-1)^2+4mn}}{2}$, $\lambda_2 = \frac{m-1-\sqrt{(m-1)^2+4mn}}{2}$.

Corollary 4.2. *If* $R = \prod_{i=1}^{k} R_i$ *be a direct product of finite commutative local rings* R_i *with* m_i *nilpotents and* n_i *units in the ring* R_i *for* $i = 1, 2, \ldots, k$*. Then*

$$\left\{(0)^{(\sum_{i=1}^{k}(n_i-1)\frac{|R|}{|R_i|})}, ((-1)^k)^{(\prod_{i=1}^{k}(m_i-1))}\right\} \subseteq \sigma_A(\Gamma'_e(R)). \tag{4.2}$$

Proof. By Theorem 4.1,

$$\sigma_A(\Gamma'_e(R_i)) = \{0^{(n_i-1)}, (-1)^{(m_i-1)}, \lambda_{i1}, \lambda_{i2}\},$$

where $\lambda_{i1}, \lambda_{i2}$ are roots of equation $x^2 - (m_i - 1)x - m_i n_i = 0$. Since $\Gamma'_e(R) = \Gamma'_e(R_1) \otimes \Gamma'_e(R_2) \otimes \cdots \otimes \Gamma'_e(R_k)$, $\sigma_A(\Gamma'_e(R))$ is a multiset and it is product of multisets $\sigma_A(\Gamma'_e(R_1)), \sigma_A(\Gamma'_e(R_2)), \ldots, \sigma_A(\Gamma'_e(R_k))$. Therefore 0 is an eigenvalue of $\sigma_A(\Gamma'_e(R))$ with multiplicity $\sum_{i=1}^{k}(n_i - 1)\frac{|R|}{|R_i|}$. Also, $(-1)^k$ is an eigenvalue with multiplicity $\prod_{i=1}^{k}(m_i - 1)$. Hence $\left\{(0)^{(\sum_{i=1}^{k}(n_i-1)\frac{|R|}{|R_i|})}, ((-1)^k)^{(\prod_{i=1}^{k}(m_i-1))}\right\} \subseteq \sigma_A(\Gamma'_e(R))$.

Corollary 4.3. *If* R *is a finite commutative local ring, then*

$$\sigma_A(\Gamma'(R)) = \left\{(-1)^{(f(R)-1)}, (f(R)-1)^{(1)}\right\}. \tag{4.3}$$

In particular,

$$\sigma_A(\Gamma'(\mathbb{Z}_{p^\alpha})) = \left\{(-1)^{(p^{\alpha-1}-1)}, (p^{\alpha-1}-1)^{(1)}\right\}.$$

Proof. If R is a finite commutative local ring, then all vertices in $\Gamma'(R)$ are nonzero nilpotent elements. Therefore, $\Gamma'(R) = K_{f(R)}$. Hence proof.

Definition 4.4. *Let* H *be a graph obtained from* $\Gamma'(S^k)$ *by merging all nilpotents into a single vertex say* **0** *and by edge contraction.*

First, we express $\Gamma'(R)$ as an H-generalized join of a complete graph and a family of null graphs.
Let

$$\chi(x_i) = \begin{cases} 1 & \text{if } x_i \text{ is unit,} \\ 0 & \text{if } x_i = 0, \\ 2 & \text{if } x_i \text{ is nonzero nonunit.} \end{cases}$$

and

$$\mathbf{C}(x) = (\chi(x_1), \chi(x_2), \ldots, \chi(x_k)) \in S^k, \text{ for each } x = (x_1, x_2, \ldots, x_k) \in R.$$

We define

$$X_0 = \left\{x \in R \colon \mathbf{C}(x) \in \{0,2\}^k \setminus \{0\}^k\right\}, \tag{4.4}$$

$$X_e = \left\{x \in R \colon \mathbf{C}(x) = e \in S^k \setminus (\{0,2\}^k \cup \{1\}^k)\right\}. \tag{4.5}$$

The family of sets

$$\left\{X_0, X_e \colon e \in S^k \setminus (\{0,2\}^k \cup \{1\}^k)\right\}. \tag{4.6}$$

These $3^k - 2^k$ sets forms a partition of the vertex set of $\Gamma'(R)$.

The following result gives the total number of nonzero nilpotent elements and the number of non-nilpotent zero-divisors in the direct product of finite commutative local rings.

Theorem 4.5. *Let $R = \prod_{i=1}^{k} R_i$, where R_i are finite commutative local rings for $i = 1, 2, \ldots, k$. Let β and γ denote the total number of nonzero nilpotent elements and the number of non-nilpotent zero-divisors in a ring R. Then*

$$\beta = |X_0| = f(R), \gamma = \sum_{e \in H \setminus \{0\}^k} |X_e|, \text{ where} \tag{4.7}$$

$$|X_e| = \phi\left(\prod_{e_i(i)=1} R_i\right) \times \prod_{e_i(1)=2} f(R_i). \tag{4.8}$$

Proof. Since ϕ is a multiplicative function, the proof follows from the multiplication principle of combinations.

In the following theorem, we express the graph $\Gamma'(R)$ as an H-generalized join graph of a complete graph and null graphs.

Theorem 4.6. *Let* $R = \prod_{i=1}^{k} R_i$ *be a direct product of finite commutative local rings and* $\mathcal{F} = \{\Gamma'(X_{\mathbf{0}}), \Gamma'(X_e) : e \in H\}$ *is the family of subgraphs of* $\Gamma'(R)$*, where*

$$X_{\mathbf{0}} = \left\{x \in R \colon \chi(x) \in \{0,2\}^k \setminus \{0\}^k\right\}, \tag{4.9}$$
$$X_e = \left\{x \in R \colon \chi(x) = e \in S^k \setminus (\{0,2\}^k \cup \{1\}^k)\right\}. \tag{4.10}$$

Then

$$\Gamma'(X_{\mathbf{0}}) = K_{|X_{\mathbf{0}}|}, \Gamma'(X_e) = \overline{K}_{|X_e|}, \tag{4.11}$$

and $\Gamma'(R)$ *is a* H*-generalized join of family* $\mathcal{F}$ *of graphs.*

Proof. Let $x \in X_e$ and $y \in X_f$. Suppose e and f are not adjacent in the graph H. Then $e^n f \neq 0$ and $ef^n \neq 0$, for any positive integer n. Hence there is $t \in \{1, 2, \ldots, k\}$ such that $e(t) = f(t) = 1$ and hence x_t and y_t both are units. Therefore $x^n y \neq 0$ and $xy^n \neq 0$, for any positive integer n. Hence, x and y are not adjacent. Also, if e and f be adjacent, then $e^2 f = 0$ or $ef^2 = 0$. There exists a positive integer n such that $x^n = e^2$ and $y^n = f^2$. Therefore, $x^n y = 0$ or $xy^n = 0$ for some positive integer n. Hence, since e and f are adjacent, it follows that x and y are adjacent. Hence, any two vertices $x \in X_e$ and $y \in X_f$ are adjacent if and only if e and f are adjacent in H. Thus $\Gamma'(R)$ is H-generalized join of family of graphs $\mathcal{F}$.

The following theorem gives an expression for the adjacency spectrum of $\Gamma'(R)$ for a direct product of finite commutative local rings.

Theorem 4.7. *Let* $R = \prod_{i=1}^{k} R_i$ *be direct product of finite commutative local rings and* $N = diag(|X_e|)_{e \in H}$*. Then*

$$\sigma(A(\Gamma'(R))) = \left\{(-1)^{(\beta-1)}, 0^{(\gamma-(3^k-2^k-1))}\right\} \cup \sigma(NA(H)). \tag{4.12}$$

Proof. By Theorem 4.6, we have

$$\Gamma'(R) = \bigvee_{H} \mathcal{F}, \tag{4.13}$$

where $\mathcal{F}$ is a family of graphs given in Theorem 4.6. Hence by Theorem 1.1, we have

$$\sigma(\Gamma'(R)) = \left\{(-1)^{(|X_{\mathbf{0}}|-1)}\right\} \cup \bigcup_{e\in H\setminus\{\mathbf{0}\}} \left\{(0)^{(|X_e|-1)}\right\} \cup \sigma(NA(H))). \qquad (4.14)$$

Therefore

$$\sigma(\Gamma'(R)) = \left\{(-1)^{|X_{\mathbf{0}}|-1}\right\} \cup \left\{(0)^{(\sum_{e\in H\setminus\{\mathbf{0}\}}(|X_e|-1))}\right\} \cup \sigma(NA(H))). \qquad (4.15)$$

By using equation (4.7), we get the expression in equation (4.12).

Theorem 4.8. *Let* $R = \prod_{i=1}^{k} R_i$ *be a direct product of finite commutative local rings* R_i. *Let* X_0 *be set of all nilpotents in* R *and for* $i = 1, 2, \ldots, k-1$, $j = 0, 1, \ldots, k-i$, *if*

$$X_{ij} = \left\{x \in R\colon \mathbf{C}(x) = e \in S^k \text{ has exactly } i\ 1's \text{ and } j\ 0's\right\}.$$

Then $X_{ij} = X_{e_1^{ij}} \cup X_{e_2^{ij}} \cup \cdots \cup X_{e_{m_{ij}}^{ij}}$, $d_1^{ij} = |X_{e_1^{ij}}|, d_2^{ij} = |X_{e_2^{ij}}|, \ldots, d_{m_{ij}}^{ij} = |X_{e_{m_{ij}}}|$, *where* $m_{ij} = \binom{k-1}{i}\binom{k-i+1}{j}$. *Let us order sets* X_{ij} *according to their increasing height, where height of* X_{ij} *is* $i+j$. *That is* $S_0 = X_0$, $S_1 = X_{11}$, $S_2 = X_{12}$, $S_3 = X_{21}$, $S_4 = X_{13}$, $S_5 = X_{22}$, $S_6 = X_{31}, \ldots$. *Let* A_{rt} *be a matrix whose rows are indexed by vertices in* S_r *and columns are indexed by vertices in* S_t *and* $D_{ij} = diag(d_1^{ij}, \ldots, d_{m_{ij}}^{ij})$ *then nontrivial eigenvalues of adjacency matrix* $A(\Gamma'(R))$ *is that of* $\sigma(NA(H))$ *and* $NA(H)$ *is a block matrix* $diag(D_{ij})[A_{ij}]$. *This block partitioning of* $N(A(H))$ *is equitable if and only if* R_i *are same for all* $i = 1, 2, \ldots, k$.

Proof. Observe that the sum of each row of A_{ij} is the same. Hence the block partition of $A(H)$ as $[A_{ij}]$ is equitable. Also, each D_{ij} is a scalar matrix if and only if R_i is the same for all i. Hence, each block of $diag(D_{ij})[A_{ij}]$ has a constant row sum. Hence proof.

Remark 4.9. *Above theorem says that if ring* $R = \prod_{i=1}^{k} R_{p^\alpha}$, *where* R_{p^α} *is a local commutative ring of order* p^α *then nontrivial eigenvalues of* $A(\Gamma'(R))$ *are given by a quotient matrix of size* $\binom{k}{2}$, *which is much smaller than that of* $A(H)$.

5. Application

Let $R_i = \mathbb{Z}_{q_i}$ be commutative local ring of order $q_i = p_i^{\alpha_i}$ for each $i = 1, 2, 3$ and $R = \prod_{i=1}^{k} R_i$. Let $N(R_i)$ denotes the set of all nilpotent elements in the ring R_i,

$N^*(R_i) = N(R_i) \setminus \{0\}$ and $U(R_i)$ denotes units in the ring R_i. In this case $\phi(R)$ becomes $\phi(|R|)$- Eulers' phi function and $f(R) = f(|R|) = |R| - \phi(|R|) - 1$. Next, we will find the adjacency spectrum of a ring R for $k = 1, 2, 3$.

(i) Let $k = 1, R = R_1$. It is clear that $\Gamma'(R_1) = K_{q_1-1}$. By Corollary 4.3,

$$\sigma_A(\Gamma'(R_1)) = \{(-1)^{(p_1^{\alpha_1-1}-2)}, (p_1^{\alpha_1-1} - 2)^{(1)}\}.$$

(ii) Let $k = 2, R = R_1 \times R_2$. A graph H is the subgraph of $\Gamma'_e(S^2)$ with vertices $e_0 = (0,0)$, $e_1 = (0,1)$, $e_2 = (1,0)$, $e_3 = (1,2)$, $e_4 = (2,1)$. The vertex set of $\Gamma'(R)$ is partitioned into a family of sets $\mathcal{F} = \{X_{e_i} : i = 0,1,2,3,4\}$ with

$$X_{e_0} = (N(R_1) \times N(R_2)) \setminus \{(0,0)\}, X_{e_1} = \{0\} \times U(R_2),$$
$$X_{e_2} = U(R_1) \times \{0\}, X_{e_3} = U(R_1) \times N^*(R_2), X_{e_4} = N^*(R_1) \times U(R_2).$$

Therefore,

$$|X_{e_0}| = f(q_1 q_2), |X_{e_1}| = \phi(q_2), |X_{e_2}| = \phi(q_1),$$
$$|X_{e_3}| = \phi(q_1) f(q_2), |X_{e_4}| = f(q_1)\phi(q_2).$$

Since $\Gamma'(R) = \bigvee_H \left\{K_{|X_{e_0}|}, \overline{K}_{|X_{e_i}|} : i \in [4]\right\}$ with underlying graph $H = \Gamma'(\{e_0, e_1, e_2, e_3, e_4\})$ as a subgraph of $\Gamma'_e(S^2)$. Therefore

$$\sigma_A(\Gamma'(R)) = \left\{0^{(\sum_{i=1}^{4}(|X_{e_i}|-1))}, (-1)^{(|X_0|-1)}\right\} \cup \sigma(Q_5), \text{ where}$$

$$Q_5 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} |X_{e_0}| & 0 & 0 & 0 & 0 \\ 0 & |X_{e_1}| & 0 & 0 & \\ 0 & 0 & |X_{e_2}| & 0 & 0 \\ 0 & 0 & 0 & |X_{e_3}| & 0 \\ 0 & 0 & 0 & 0 & |X_{e_4}| \end{bmatrix}.$$

(iii) Let $k = 3, R = R_1 \times R_2 \times R_3$. In this case, $\Gamma'(R)$ is H-generalized join of a family of graphs $\left\{K_{|X_{e_0}|}, \overline{K}_{|X_{e_i}|} : e_i \in H_3 \setminus \{e_0\}\right\}$, where H is subgraph of $\Gamma'_e(S^3)$ on vertices $e_0 = (0,0,0)$, $e_1 = (0,0,1)$, $e_2 = (0,1,0)$, $e_3 = (0,1,1)$, $e_4 = (0,1,2)$, $e_5 = (0,2,1)$, $e_6 = (1,0,0)$, $e_7 = (1,0,1)$, $e_8 = (1,0,2)$, $e_9 = (1,1,0)$, $e_{10} = (1,1,2)$, $e_{11} = (1,2,0)$, $e_{12} = (1,2,1)$, $e_{13} = (1,2,2)$, $e_{14} = (2,0,1)$, $e_{15} = (2,1,0)$, $e_{16} = (2,1,1)$, $e_{17} = (2,1,2)$, $e_{18} = (2,2,1)$. Therefore

$$
\begin{aligned}
&|X_0| = f(q_1q_2q_3),\\
&|X_{e_1}| = \phi(q_1), |X_{e_2}| = \phi(q_2), |X_{e_3}| = \phi(q_2q_3)\\
&|X_{e_4}| = \phi(q_2)f(q_3), |X_{e_5}| = \phi(q_3)f(q_2),\\
&\vdots\\
&|X_{e_{18}}| = f(q_1)f(q_2)\phi(q_3).\\
\sigma_A(\Gamma'(R)) = &\left\{0^{\sum_{i=1}^{18}(|X_{e_i}|-1)}, (-1)^{(|X_0|-1)}\right\} \cup \sigma(Q_{19}), \text{ where}\\
&Q_{19} = A(H)diag(|X_e|)_{e\in H}.
\end{aligned}
$$

Now, according to the equation (3.17), we write an equitable partition of the vertex set of R as below.

$$X_0 = X_{e_0}, \tag{5.1}$$
$$X_{1,0} = X_{e_{13}} \cup X_{e_{17}} \cup X_{e_{18}}, \tag{5.2}$$
$$X_{1,1} = X_{e_4} \cup X_{e_5} \cup X_{e_8} \cup X_{e_{11}} \cup X_{e_{14}} \cup X_{e_{15}}, \tag{5.3}$$
$$X_{1,2} = X_{e_1} \cup X_{e_2} \cup X_{e_6}, \tag{5.4}$$
$$X_{2,0} = X_{e_{10}} \cup X_{e_{12}} \cup X_{e_{16}}, \tag{5.5}$$
$$X_{2,1} = X_{e_3} \cup X_{e_7} \cup X_{e_9}. \tag{5.6}$$

We take the sets that partition the vertex set of H as follows.
$S_0 = \{e_0\}$, $S_1 = \{e_{13}, e_{17}, e_{18}\}$, $S_2 = \{e_4, e_5, e_8, e_{11}, e_{14}, e_{15}\}$, $S_3 = \{e_1, e_2, e_6\}$, $S_4 = \{e_{10}, e_{12}, e_{16}\}$, $S_5 = \{e_3, e_7, e_9\}$.
Suppose that A_{ij} is a submatrix of $A(H)$ whose rows are indexed with vertices in S_i and columns are indexed by S_j. Then $A(H) = [A_{ij}]_{i=0,j=0}^{5,5}$ and note that $A_{ij} = A_{ji}^t$. Observe that

$$A_{00} = O_1,\ A_{11} = A_{55} = O_3,\ A_{33} = \begin{bmatrix} 0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0 \end{bmatrix},\ A_{44} = O_3, A_{55} = O_3,$$

$$A_{22} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0\\ 0 & 0 & 1 & 1 & 0 & 1\\ 1 & 1 & 0 & 0 & 0 & 1\\ 1 & 1 & 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 1 & 0 & 1\\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix},\ A_{01} = A_{03} = A_{04} = A_{05} = 1_{1,3}, A_{02} = 1_{1,6},$$

$$A_{12} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix},\ A_{13} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},\ A_{14} = O_3,\ A_{15} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

$$A_{32} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix},\ A_{42} = O_{3,6},\ A_{52} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$A_{34} = I_3,\ A_{35} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ A_{45} = O_3.$$

Let $N_3 = diag(D_0, D_1, D_2, D_3, D_4, D_5)$. Then $Q_{19} = N_3 A(H_3)$ and diagonal matrices $D_0, D_1, D_2, D_3, D_4, D_5$ are given by

$$D_0 = f(q_1q_2q_3),$$
$$D_1 = diag(\phi(q_1)f(q_2)f(q_3), f(q_1)\phi(q_2)f(q_3), f(q_1)f(q_2)\phi(q_3)),$$
$$D_2 = diag(\phi(q_2)f(q_3), f(q_2)\phi(q_3), \phi(q_1)f(q_3), f(q_1)\phi(q_3), \phi(q_1)f(q_2), f(q_1)\phi(q_2)),$$
$$D_3 = diag(\phi(q_3), \phi(q_2), \phi(q_1)),$$
$$D_4 = diag(\phi(q_1q_2)f(q_3), \phi(q_1q_3)f(q_2), f(q_1)\phi(q_2q_3)),$$
$$D_5 = diag(\phi(q_2q_3), \phi(q_1q_3), \phi(q_1q_2).$$

The block diagonal matrix $[A_{ij}]_{i=0,j=0}^{5,5}$ forms an equitable partition of the matrix $A(H)$ and corresponding quotient matrix is

$$Q = \begin{bmatrix} 0 & 3 & 6 & 3 & 3 & 3 \\ 1 & 0 & 2 & 2 & 0 & 1 \\ 1 & 1 & 3 & 2 & 0 & 1 \\ 1 & 2 & 3 & 3 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 \end{bmatrix} = [q_{ij}].$$

The matrix Q_{19} can be written into block diagonal form as $[D_iA_{ij}]_{i=0,j=0}^{5,5}$. Its quotient matrix is $Q' = \left[trace(D_i)\frac{q_{ij}}{size(A_{ij})}\right]_{i=0,j=0}^{5,5}$. Eigenvalues of Q' interlace eigenvalues of Q_{19}.

6. Conclusion

Expression (4.12) in Theorem 4.7 gives multiplicities of eigenvalues -1 and 0 in terms of the number of nonzero nilpotent elements and non-nilpotent zero-divisors, respectively, in the ring. The remaining eigenvalues are in terms of the eigenvalues of $\Gamma'(S^k)$. Many authors expressed zero-divisor graphs as a generalized join of other graphs and obtained the properties of zero-divisor graphs. In expression (4.13), we have expressed the graph $\Gamma'(R)$ as a multi-partite graph with one component a null graph, and the remaining components are all complete graphs. Therefore, if $k \leq 3$ and $p_i^{\alpha_i - 1} \leq 4$, then $\Gamma'(R)$ is a planar graph. Many other structural properties of the graph $\Gamma'(R)$ can be observed from the expression (4.13).

Acknowledgements

The authors are deeply thankful to anonymous referees for their valuable comments and suggestions, which helped to improve the presentation of the paper.

References

[1] Anderson D. F. and Livingston P., The zero-divisor graph of a commutative ring, J. Algebra, 217 (1999), 434-447.

[2] Anderson D. F., Asir T., Badawi A., and Chelvam T., Graphs from rings, Springer, 2021.

[3] Atiyah M., Introduction to commutative algebra, (CRC Press), 2018.

[4] Atik Fouzul, On equitable partition of matrices and its applications, Linear and Multilinear Algebra, 68(11) (2019), 2143–2156.

[5] Beck I., Coloring of commutative rings, J. Algebra, 116 (1988), 208-226.

[6] Cardoso D., Freitas M., Martins E., and Robbiano M., Spectra of graphs obtained by a generalization of the join graph operation, Discrete Math., 313 (2013), 733-741.

[7] Cardoso D. M. and Rama P. C., Equitable partitions of graphs and related results, J. Math. Sci., 120 (2004), 869-880.

[8] Godsil C. and Royle G., Algebraic graph theory, Springer Science and Business Media, 2001.

[9] Kumbhar N., Khairnar A. and Waphare B. N., Strong zero-divisor graph of rings with involution, Asian-Eur. J. Math., 16(10) (2023), #2350179.

[10] Lagrange J. D., Eigenvalues of Boolean graphs and Pascal type matrices, Int. Electron. J. Algebra, 13 (2013), 109-119.

[11] Lande A. and Khairnar A., Generalized zero-divisor graph of $*$-rings, arXiv:2403.10161v1 [math.CO] (2024).

[12] Lande A. and Khairnar A., On the spectrum of generalized zero-divisor graph of the ring $\mathbb{Z}_{p^\alpha q^\beta}$, Communications in Mathematics and Applications, 15(3) (2024), 1031-1044.

[13] Lande A. and Khairnar A., Idempotent graph of 2×2 matrix ring with involution, Gulf J. Math., 19(2) (2025), 168-180.

[14] Lande A., Khairnar A., and Gutman I., The zero-divisor graph of 2×2 matrix ring and its energies, Filomat, 39(22) (2025).

[15] Patil A. and Waphare B., The zero-divisor graph of a ring with involution, J. Algebra Appl., 17(03) (2018), #1850050.

[16] Pirzada S., Wani B., and Somasundaram A., On the eigenvalues of zero-divisor graph associated to finite commutative ring, AKCE Int. J. Graphs Comb., 18 (2021), 1-6.

[17] Saravanan M., Murugan S. P., and Arunkumar G., A generalization of Fiedler's lemma and the spectra of H-join of graphs, Linear Algebra Appl., 625(15) (2021), 20-43.

[18] Sonawane G., Kadu G. and Borse Y., Spectra of zero-divisor graphs of finite reduced rings, J. Algebra Appl., 24(3) (2025), # 2550082.

[19] You Lihua, Yang Man, So Wasin, and Xi Weige, On the spectrum of an equitable quotient matrix and its application, Linear Algebra Appl., 577 (2019), 21-40.